GOLDBACH RADIUS

by Alex Milona, Gabe Follettie & Selim Tezel

On June 7, 1742 Prussian mathematician Christian Goldbach wrote a letter to fellow mathematician Leonhard Euler asking for help in proving a conjecture. In his letter, Goldbach stated, "at least it seems that every number that is greater than two is the sum of three primes" (Goldbach 1742; Dickson 2005, p. 421).[1] Goldbach considered one to be a prime number when he wrote this letter to Euler, though mathematicians have since abandoned that convention. Euler reiterated Goldbach's question in his reply by postulating that "all positive even integers greater than or equal to four are expressible as the sum of two prime numbers". Today, Goldbach's conjecture is expressed in the following form: every even integer *e* greater than two is the sum of two primes. This conjecture is written mathematically below.

Goldbach Conjecture

For all even integers e >2, there exist primes P_1 , P_2 such that $e = P_1 + P_2$

Though he found a means to restate the conjecture, the great Euler could make no headway in the proof. He replied, "I regard this as a completely certain theorem, though I have no way to prove it."[2] The proof of the Goldbach conjecture remains an open question for mathematicians to this day.

Statistics teacher Selim Tezel introduced us to Goldbach's conjecture while we worked on other statistics-related problems utilizing the statistics software Fathom 2. The goal was to utilize computational software in an analysis of the "Goldbach radius," a concept, which will be defined later, first developed by CA students Bronwyn Murray-Bozeman '10, and Lizzie Durney '10.

Goldbach's conjecture can be related mathematically to a radius value in an elegant manner.

If e is a positive even integer, then e = 2N for some positive integer N. Therefore, Goldbach's conjecture is expressible in the form:

$$e = 2N = P_1 + P_2 \text{ simplified to the form } N = \frac{(P_1 + P_2)}{2}$$

Thus in words, an equivalent version of the Goldbach conjecture would say:

Goldbach Conjecture Variation

All positive integers greater than one are expressible as the <u>average</u> of two primes.

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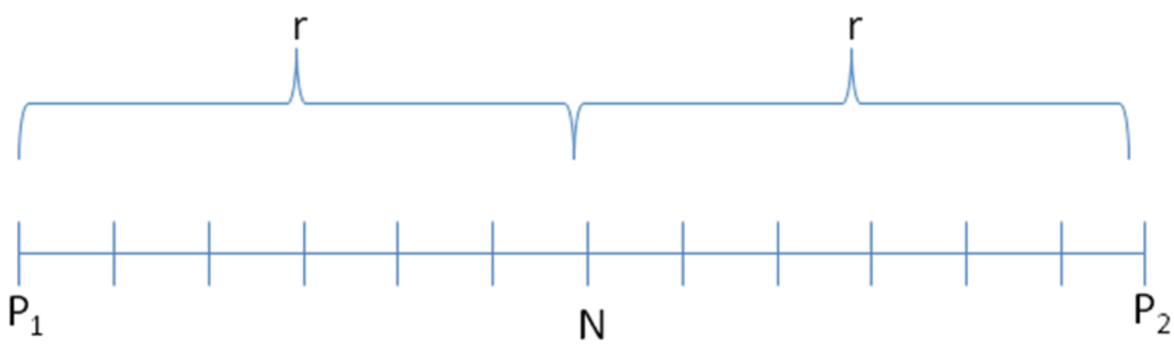
[1] Eric W.Winstein "Goldbach Conjecture." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/GoldbachConjecture.html

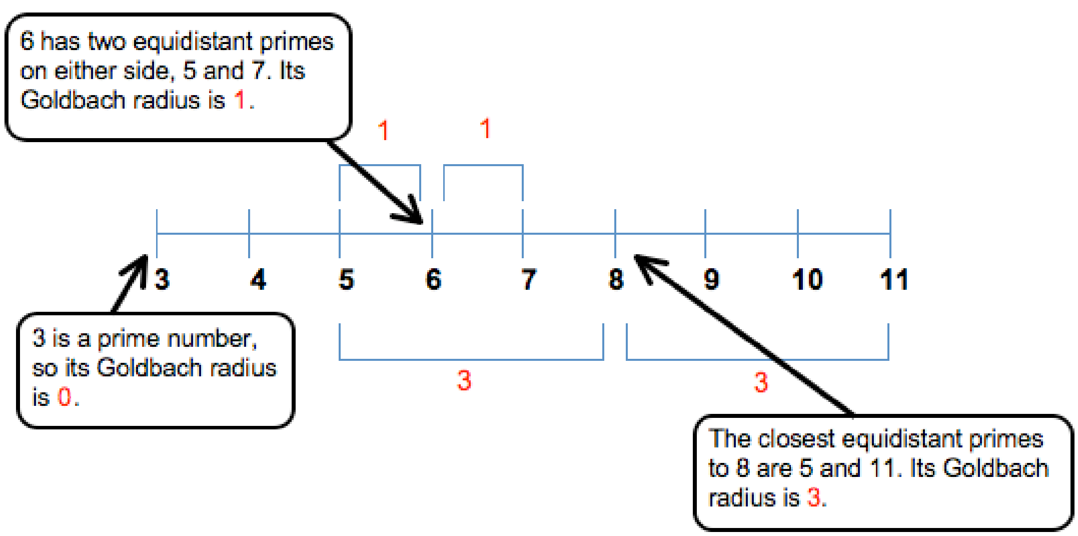
[2] Clay Mathematics Institute. "Riemann." http://www.claymath.org/Popular_Lectures/U_Texas/Riemann_1.pdf.

When graphed on a number line, this equation demonstrates the existence of two or more prime numbers equidistant to N. The distance from N to the nearest pair of equally far prime numbers we define as the "Goldbach radius."



N =Any positive integer
$P_2 = N+r$
$P_1 = N-r$
r = Goldbach radius of N

A number line with illustrated Goldbach radii is included below.

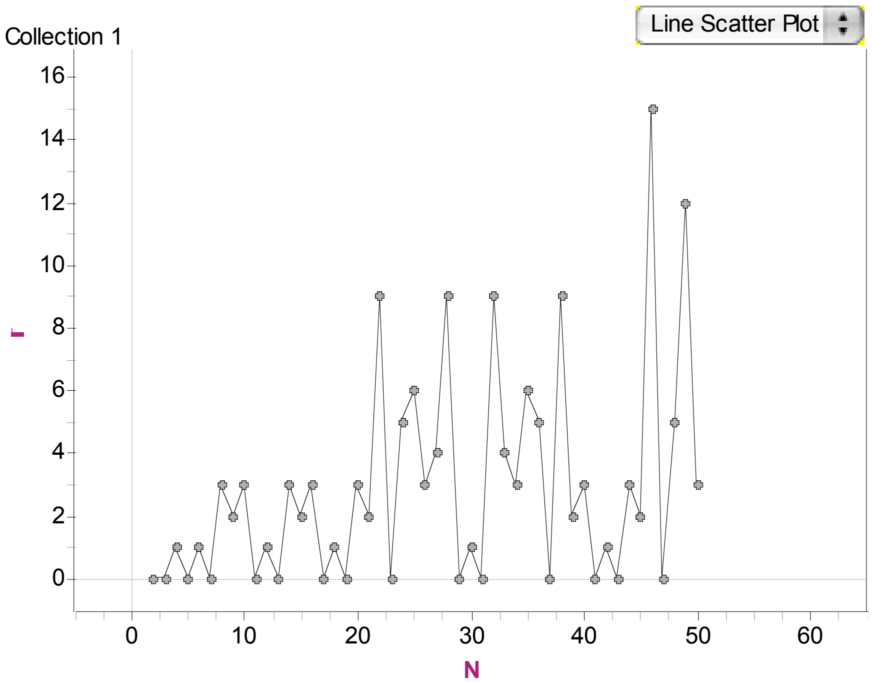


The process for producing the results seen on the subsequent page involved additional help from fellow CA mathematicians. Russell Cohen '09 wrote a computer program that generated the Goldbach radii of numbers [1,1000,000]. Indeed, it was this list, which was input into the Fathom 2 program that allowed the project to move forward.

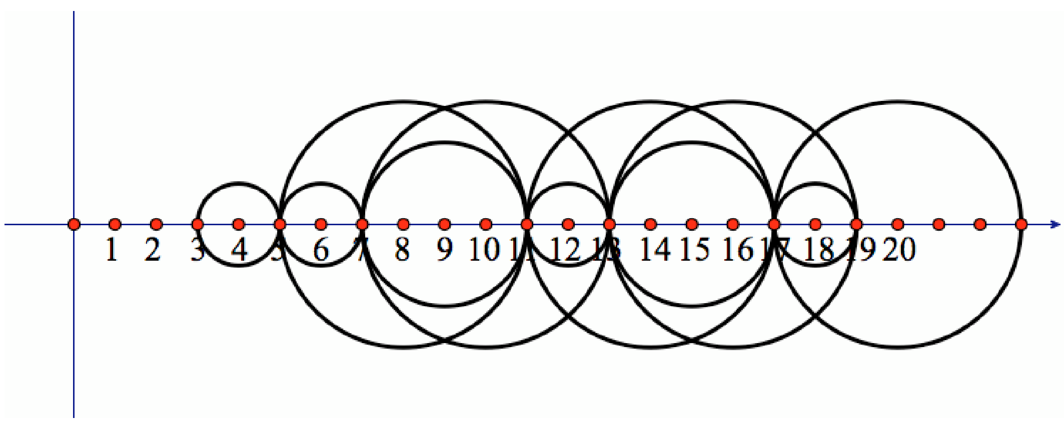
The list below shows the numbers *N* from two through forty with their corresponding Goldbach radii.

N	r: Goldbach Radius
2	0
3	0
4	1
5	0
6	1
7	0
8	3
9	2
10	3
11	0
12	1
13	0
14	3
15	2
16	3
17	0
18	1
19	0
20	3
21	2
22	9
23	0
24	5
25	6
26	3
27	4
28	9
29	0
30	1
31	0
32	9
33	4
34	3
35	6
36	5
37	0
38	9
39	2
40	3

The graph of integers N in the domain [1,50] versus their Goldbach radius r



Another representation of Goldbach radii associated with integers in domain [1,20]



After creating scatter plots recording the Goldbach radii for all numbers from [1,1000] we decided that a good first step in learning more about Goldbach's conjecture was to find patterns among the Goldbach radii by filtering the results shown above for radii divisible by two, three, four, etc. We also filtered the Goldbach radii by prime Goldbach radii. At first glance the graph of prime Goldbach radii compared to non-prime Goldbach radii demanded further investigation, as it revealed a remarkably high number of prime Goldbach radii.

So we took Russell Cohen's list and recorded the percentages of prime Goldbach Radii in 22 groups of 10,000. We choose these groups arbitrarily and tried to space out the groups as much as possible. Here is a table of our results:

Results:

Search domain *% of numbers in domain with prime Goldbach radii*

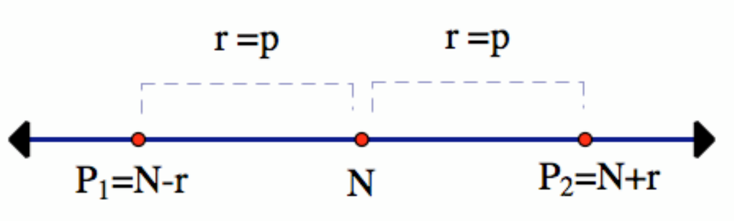
1-10,000	**18.49%**
10,000-20,000	**16.53%**
50,000-60,000	**15.25%**
100000-110000	**14.73%**
130000-140000	**14.20%**
150000-160000	**13.97%**
200000-210000	**13.79%**
260000-270000	**13.86%**
320000-330000	**13.38%**
390000-400000	**13.73%**
410000-420000	**13.44%**
460000-470000	**13.29%**
540000-550000	**13.29%**
580000-590000	**13.14%**
630000-640000	**13.09%**
680000-690000	**12.93%**
700000-710000	**13.24%**
760000-770000	**13.29%**
830000-840000	**13.27%**
850000-860000	**13.17%**
940000-950000	**12.86%**
990000-1000000	**12.92%**

This table, for example, shows that for the domain [1, 10,000], 1849 of these numbers have prime Goldbach radii. Therefore, approximately 18.49% of the numbers in the domain [1,10000] have Goldbach radii that are prime.

Note: A more thorough search in the entire domain [1, 1,000,000] reveals that of the first one million numbers, 13.58% of them have prime Goldbach radii. (See Appendix 1 for a Java program code for this exploration)

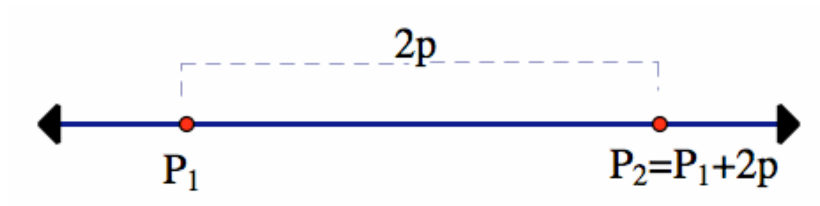
Conclusions

We observed that a surprisingly large number of Goldbach radii are prime.
For the numbers with a prime Goldbach radius we can set up the following equations:



Let N be a number with a Goldbach radius r = p, where p is a prime.
Then we can say:
$P_1 = N - r = N - p$
$P_2 = N + r = N + p$

Thus $N= P_1+p$ from the first equation and then plugging into the second equation
$P_2=N + p = (P_1 + p) + p$
We get
$P_2=P_1 + 2p$



Numbers with prime Goldbach radii reveal the existence of an interesting family of primes of the form shown above where the prime number is the sum of a prime number and double another prime number.

Definition: A prime number P is called an ***k-arithmetic prime*** if and only if it has the following skeletal form:

$P= p_1+kp_2$ where p_1, p_2 are primes and k is an integer.

Note: This name is inspired by the structure of arithmetic sequences $a_n= a_1 + kD$, where a_1 is the first term, k is an integer and the D is the common difference between the terms. For example:
$a_n= 2 + k3$ generates the arithmetic sequence 2,5,8,11,14, ….etc.

Note: All primes greater than or equal to 5 are *k-arithmetic* as their difference is always even. Thus if we let the larger of any such two primes be P and the smaller be p_1 , we have $P- p_1=2k$, and rearranging $P= p_1+k(2)$. Thus we can see that they satisfy the definition for *k-arithmetic* structure with $p_2=2$.

Definition: A prime number P is called a ***di-arithmetic prime*** if and only if it has the following skeletal form:

$$P= p_1+2p_2 \text{ where } p_1, p_2 \text{ are primes.}$$

Note: A *di-arithmetic* prime is a special *k-arithmetic* prime with k=2.

Based on the large number of integers with prime Goldbach radius we observed in the domain [1, 1,000,000] we would like to make following conjectures.

Conjecture 1: *(Prime Goldbach Radii Conjecture)*

There are infinitely many integers with prime Goldbach radius.

Note: If the previous conjecture were provable to be true then the next conjecture would follow as a corollary.

Conjecture 2: (*Di-Arithmetic Primes Conjecture)*

There exist an infinite number of primes that are di-arithmetic.

In other words, we claim that there are infinitely many primes P of the form:

$$P= p_1+2p_2 \quad \text{where } p_1, p_2 \text{ are primes.}$$

Note: Primes that differ by 4 (such as 3&7, 7&11 etc) are called "*cousin primes*" in Number Theory. There is a conjecture that states that there are infinitely many such primes. If this conjecture were proven to be true then our Conjecture 2 would in effect be proven to be true as di-arithmetic primes of the form $P= p_1+2(2)= p_1+4$ are by definition "*cousin primes*". Our conjecture however claims the existence of an infinitude of di-arithmetic primes beyond the "*cousin prime"* family.

Further Research Ideas:

-Generate a list of di-arithmetic primes in the domain [1, 1,000,000] and beyond. Find the density of these primes in the domain.

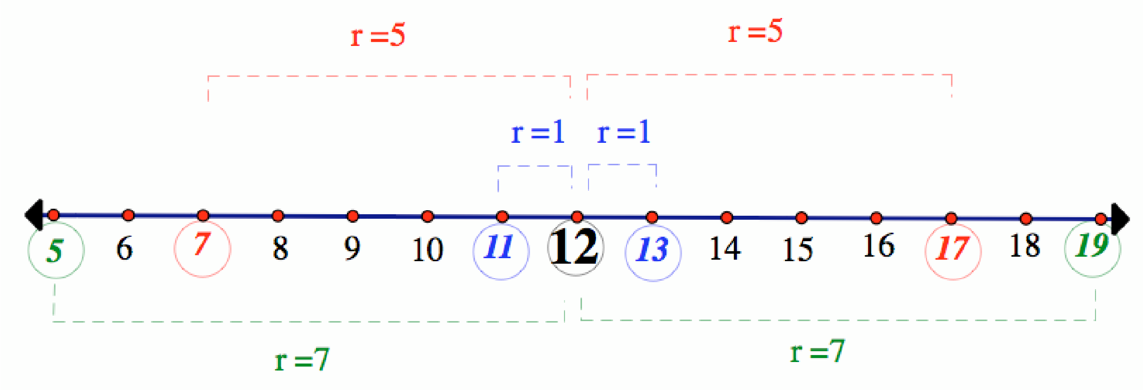
-Find the smallest prime greater than 5 that is NOT *di-arithmetic*.

-Prove or disprove the" *Prime Goldbach Radii Conjecture" and the* "*Di-Arithmetic Primes Conjecture*".

-Explore the *k-arithmetic prime* family and discover any constraints on k, in the skeletal form $P= p_1+kp_2$

-For odd integers N, it can be easily proven (see Appendix 2) that the only prime Goldbach radius allowed is 2. For even integers find out if there are any constraints on the nature of their Goldbach radii.

-In this paper we restricted the definition of Goldbach radius of an integer to be the distance from the integer to the two nearest equally far primes to the integer. Explore what would happen if we relaxed this definition (by dropping the word "nearest") and allowed an integer to have a multitude of Goldbach radii such as shown below?



Here the integer 12 (at the center) is shown to have three Goldbach radii r=1,5,7. Each of these radii leads to primes equally far from 12 on either side of the integer.

We are grateful to Concord Academy Number Theory teacher Bill Adams for his insights and suggestions for this paper.

Please contact Statistics teacher Selim Tezel at stezel@concordacademy.org for any ideas, suggestions etc. on how to further progress in these areas of research.

Appendix 1: A Java code to explore the Goldbach Radii of numbers in the domain [1, 1000,000]

```
public class GoldbachRadius {
      public static Boolean[] isPrime;
      public static int Min;
      public static int Max;

      public static void main(String[] args) {
             Min=2;
             Max=1000000;

            isPrime=new Boolean[2*Max+1];
            generatePrimes();

            int numOfPrimeGoldbachRadius=0;

            for(int n=Min;n<Max+1; n++)
            {
                  int r=getGoldbachRadius(n);
                  System.out.println(n+" "+r+" "+(n-r)+" "+(n+r));
                  if(isPrime[r])
                        {
                        numOfPrimeGoldbachRadius++;
                        };
            }

            double ratio=
(double)(100.0*numOfPrimeGoldbachRadius)/(Max-Min+1);

            System.out.println("percentage of prime Goldbach
radii:"+ratio+"%");


      }
```

```
public static void generatePrimes()
          { //initialize
               for(int i=0; i<2*Max+1;i++)
               {
                    isPrime[i]=true;
               };
               isPrime[0]=false;
               isPrime[1]=false;
               //Sieve of Eratosthenes
               int i=2;
               while(i<2*Max+1)
               {
                    if(isPrime[i]==true)
                    {
                         int k=2;
                         while(i*k<2*Max+1)
                         {
                              isPrime[i*k]=false;
                              k++;
                         };
                    };
                    i++;
                    }
          }

          public static int getGoldbachRadius(int N){
               boolean done=false;
               int r=0;
               while(!done && r<=N)
               {
                    if(isPrime[N-r] && isPrime[N+r])
                    {
                         done=true;
                    }
                    else
                         r++;
               }
               if (done==true)
                    return r;
               else
                    return -1;
          }
          }
```

Appendix 2: Goldbach radius for odd integers

Definition: *A theorem is to be called a "**Goldbach Conjecture Dependent**" (**GCD**) **Theorem** if and only if its correctness depends of on the correctness of the (yet unproven) Goldbach Conjecture.*

GCD Theorem: (depends on the truth of the Goldbach Conjecture)
If N is an odd integer greater than 1, then its Goldbach radius is even.

Proof:
By the variation of the Goldbach conjecture we know that for any integer N >1 there exist two primes P_1, P_2 such that N is their average:
N = $\frac{(P_1 + P_2)}{2}$ and thus P_1 = N - r and P_2 = N + r , where r is the Goldbach radius for N.
If N is an odd number greater than 1 then it must be greater than or equal to 3 and then P_2 must be greater than or equal to 3 and it must be an odd prime as the only even prime is 2.
So solving for r, the Goldbach radius in the equation above we get:
r =P_2 - N
r: odd - odd = even

GCD Corollary: (depends on the truth of the Goldbach Conjecture)
If N is an odd integer greater than 1 with a prime Goldbach radius, then this radius must be 2.

Proof:
By the previous (Goldbach Conjecture dependent) Theorem we know that for such an odd integer N, the Goldbach radius is even. The only even prime is 2 so this prime Goldbach radius must be 2.

Note: A thorough computational search in the entire domain [1, 1,000,000] reveals that of the first one million numbers, only 0.8143% of them have a Goldbach radius of 2.