Representation of Higher Dimensions in 2D

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Introduction:

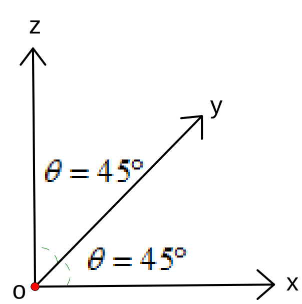
This paper explores the development of a system of mathematical equations to represent higher dimensions in 2 dimensions (2D). In order to succeed with n-dimensional mapping into 2D, one must give up one's "obsession" with 90° between axes. Then, one can map multiple dimensions into 2 dimensions by using trigonometric relationships between the axes that are used to define the location of a point. To demonstrate this concept, this paper applies the system of mathematical equations derived to construct 4-dimensional and 5-dimensional "cubes."

N-Dimensional Concept Analysis:

To begin the analysis, we have to define what we mean by a "dimension." A dimension is one coordinate in a set of n coordinates used to define the location of a point.

Start with visualizing a 2-dimensional system, the familiar X vs. Y coordinate system. Everyone is also familiar with the representation of 3 dimensions in 2 dimensions, which is done by adding a third axis (z-axis) which people can view in perspective (i.e. relate to the 3 sides of a cube from a corner) because they have been conditioned to relate it to their perception of the real physical world around them. We live in a 3-dimensional world with spaces and volumes and we are trained at an early age to represent 3 dimensions in 2-dimensional drawings as shown in Figure 1. Mathematically, the y axis in Figure 1 divides the 90° defined by the z and x axes, creating an angle θ (theta) with the x axis. By setting $\theta = 45°$, one makes the graphical representation to appear balanced and simplifies the mathematical calculations that follow below.

Figure 1
3D Axes in 2D

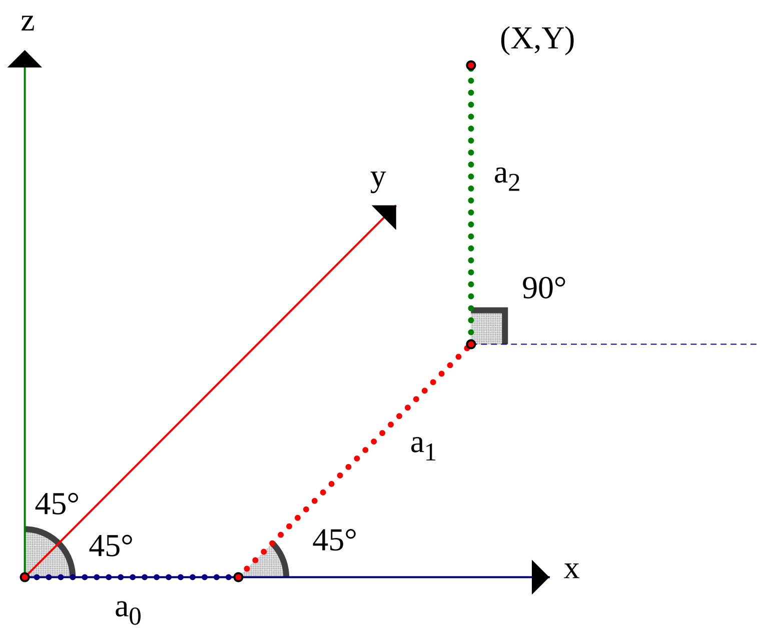


You can identify any point in 3 dimensions on a 2-dimensional graph by using trigonometric formulas to calculate X and Y coordinates. As demonstrated in Figure 2, One can map a point defined with an x coordinate value of a_0, a y coordinate of a_1, and a z value of a_2 with the following equations deriving the "mapped" x and y coordinates in a 2D graph. The x axis is used as the initial reference axis and the y and z axes' coordinates are defined by angles 45° and 90° as well as parameters a_1 and a_2.

Formulas to represent the coordinates (a_0, a_1, a_2) of a 3D point in 2D:

$$X = a_0 \cos(0°) + a_1 \cos(45°) + a_2 \cos(90°)$$
$$Y = a_0 \sin(0°) + a_1 \sin(45°) + a_2 \sin(90°)$$

Figure 2: 3D Axes in 2D Depicting Plotted Point



Mathematically, this n-dimensional mapping into 2D does not have to be limited to 3 dimensions. By defining "dimensions" as a set of coordinates to define points a certain distance from a starting point (defined as (0,0,0,…)), and by extending the logic used previously, one can define n dimensions by subdividing the original 2-dimensional 90° angle into (n-1) angles to accommodate (n-2) additional axes. The resulting formulas to find every point in a 2 dimensional X-Y plane, defining an n-dimensional point $(a_0, a_1, a_2 \ldots a_{n-1})$ are:

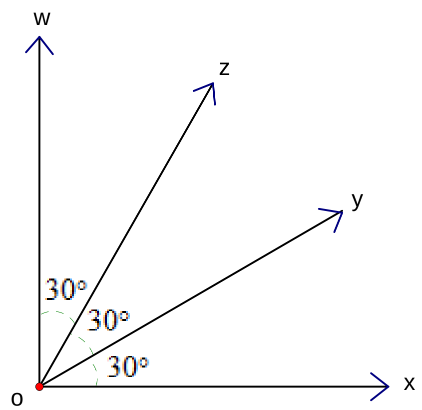
Formulas to represent n dimensions in 2D:

$$X = a_0 \cos(0°) + a_1 \cos\left(\frac{90°}{n-1}\right) + a_2 \cos\left(\frac{2*90°}{n-1}\right) + ...a_{n-1} \cos\left(\frac{(n-1)*90°}{n-1}\right) = \sum_{i=0}^{n-1} a_i \cos\left(\frac{i*90°}{n-1}\right)$$

$$Y = a_0 \sin(0°) + a_1 \sin\left(\frac{90°}{n-1}\right) + a_2 \sin\left(\frac{2*90°}{n-1}\right) + ...a_{n-1} \sin\left(\frac{(n-1)*90°}{n-1}\right) = \sum_{i=0}^{n-1} a_i \sin\left(\frac{i*90°}{n-1}\right)$$

By applying these concepts/formulas to 4 dimensions, one can construct a 4-dimensional cube. First, we define two new axes that are equal 30° degrees apart from each other and the original 2D axes, as shown in Figure 3:

Figure 3
4D Axes in 2D



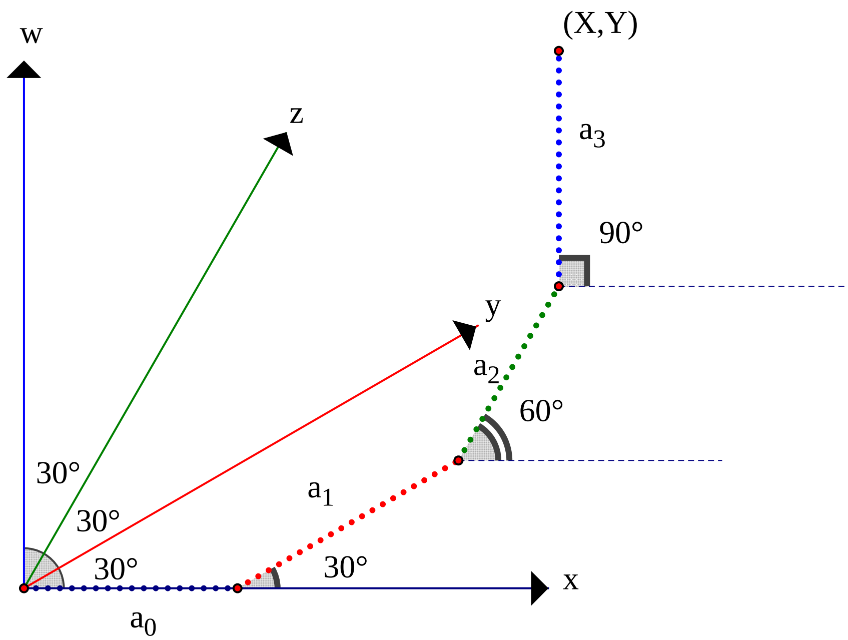
The formulas for plotting any 4-dimensional point (a_0, a_1, a_2, a_3) in 2D are:

$$X = a_0 \cos(0°) + a_1 \cos(30°) + a_2 \cos(60°) + a_3 \cos(90°)$$

$$Y = a_0 \sin(0°) + a_1 \sin(30°) + a_2 \sin(60°) + a_3 \sin(90°)$$

The plotting of a 4-dimensional point is shown in Figure 4.

Figure 4: 4D Axes in 2D Depicting Plotted Point



Representing N-Dimensional "Cubes" in 2D

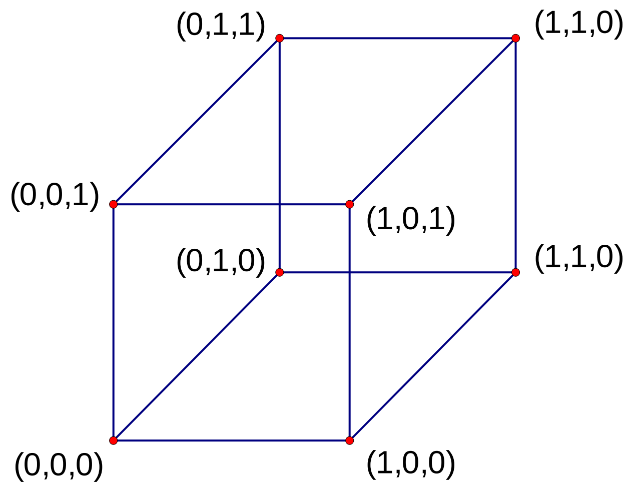
Having developed the mathematical model above, one can apply it to mathematical constructs incorporating multiple points on each of the n axes. The simplest such n-dimensional multipoint construct would be 3 dimensions, with 2 points on each axis. One could find all the points by using the formulas:

$$X = a_0 \cos(0°) + a_1 \cos(45°) + a_2 \cos(90°)$$

$$Y = a_0 \sin(0°) + a_1 \sin(45°) + a_2 \sin(90°)$$

One can define 2 points on each axis by allowing a_i to take the values of 0 and 1. This creates a unit length of 1 along each dimension. With 3 parameters, each with 2 values, one ends up plotting $2^3 = 8$ unique points [(0,0,0), (0,0,1), (0,1,0), (1,0,0), (0,1,1), (1,1,0), (1,0,1), (1,1,1)] in the X-Y plane as shown in Figure 5.

Figure 5
3 Dimensional Cube in 2D



The 3 dimension, 2 parameter values plotted in Figure 5 results in a cube with 8 vertices (the mapped 3D points) and the 8 cube edges with a length of 1 (the unit length on each axis created by allowing the 3 parameters (a_0, a_1, a_2) to take the values of 0 and 1).

By applying the concept to more than 3 dimensions, one can ponder the mathematical existence of an n-dimensional "cube", which can be represented in 2D because we have already shown above that you can plot any n-dimensional point in 2D. Therefore, you can plot the 2^n unique vertices of an n-dimensional "cube" by allowing a_i to take the values of 0 and 1 in the formulas:

$$X = a_0 \cos(0°) + a_1 \cos\left(\frac{90°}{n-1}\right) + a_2 \cos\left(\frac{2*90°}{n-1}\right) + \ldots a_{n-1} \cos\left(\frac{(n-1)*90°}{n-1}\right) = \sum_{i=0}^{n-1} a_i \cos\left(\frac{i*90°}{n-1}\right)$$

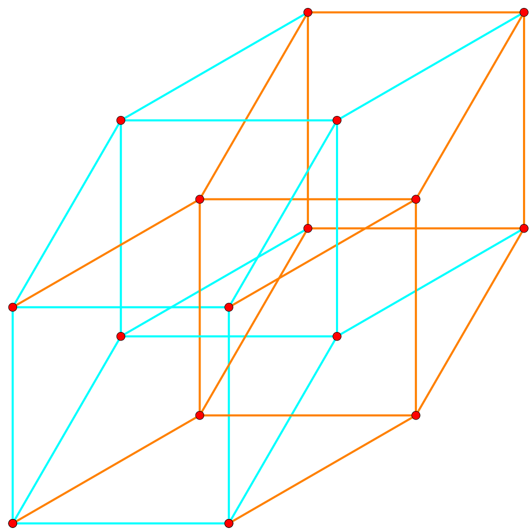
$$Y = a_0 \sin(0°) + a_1 \sin\left(\frac{90°}{n-1}\right) + a_2 \sin\left(\frac{2*90°}{n-1}\right) + \ldots a_{n-1} \sin\left(\frac{(n-1)*90°}{n-1}\right) = \sum_{i=0}^{n-1} a_i \sin\left(\frac{i*90°}{n-1}\right)$$

Therefore, as a mathematical concept, one can define an n-dimensional "cube" (where $n \geq 3$) represented in 2D, as a mathematical system defined by:

1. the X and Y values derived from the above formulas
2. a_i takes the values of 0 and 1, creating a unit length of 1 for each "edge" of the "cube." By finding the sum of the coordinates, one can find the points adjacent to a specific point. For example, point (0,0,0) is adjacent to points (0,0,1), (0,1,0), and (1,0,0) because the sum of the coordinates differ by 1.
3. 2^n vertices

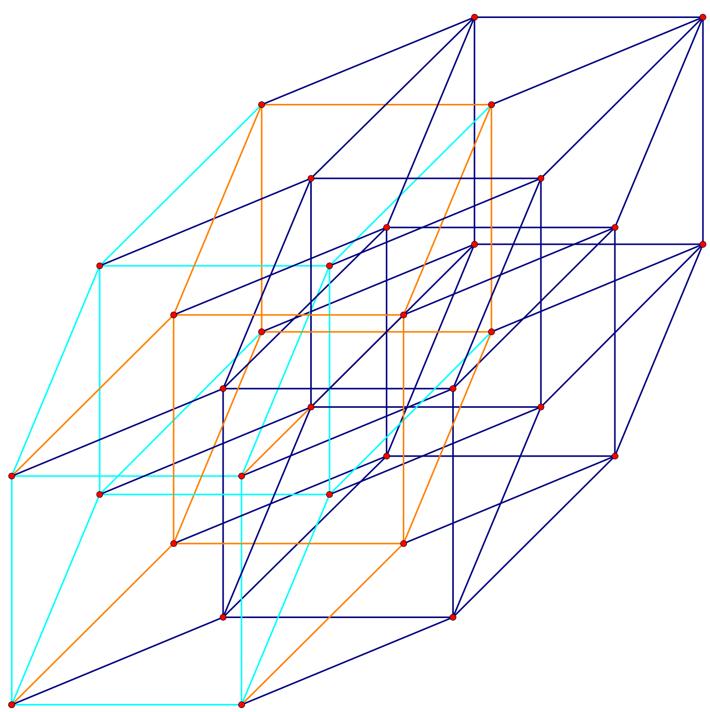
The model was applied to create a 4 dimensional "cube" with $2^4 = 16$ vertices, as shown in Figure 6.

Figure 6
4D Cube in 2D



Similarly, a 5D "cube" was created with $2^5 = 32$ vertices, as shown in Figure 7.

Figure 7
5D Cube in 2D



Conclusion:

There are two primary conclusions from this analysis:

1. After giving up one's "obsession" with 90° between the axes, an n-dimensional point $(a_0, a_1, a_2\ldots, a_{n-1})$ can be plotted in 2 dimensions using the trigonometric formulas shown above to derive the X and Y coordinates.
2. N-dimensional cubes can be represented in 2D with X and Y values defined by trigonometric functions for each unique point representing a vertex of the "cube." The analysis above demonstrates the methodology from plotting the 2^n vertices of an n-dimensional cube in 2D. By allowing the parameters $(a_0, a_1, a_2\ldots a_{n-1})$ to take values $a_i = 0$ and $a_i = 1$, you can plot any vertex of the cube.

I invite readers to find out if there are natural problems /applications where these concepts and formulas can help.