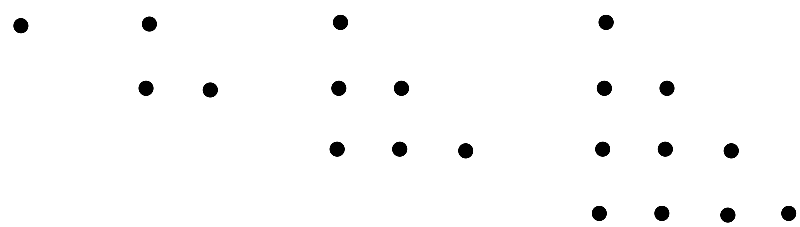
Geometric Construction of Reciprocal Triangular Numbers

via a Study of Perspective Drawing and a Demonstration That Their Sum is 2.

by Selim Tezel, in collaboration with Burak Cendek, Berkin Ozisikyilmaz and Kerem Celikel of Istanbul American Robert College.

The illustrious career of the great mathematician Gottfried Leibniz supposedly began to flourish when in 1672 (at the age of 26) he came into contact in Paris with the great Dutch scientist Christiaan Huygens. Originally sent to France on a diplomatic mission, Leibniz took the opportunity to broaden his scientific and mathematical horizons and, under the guidance and advice of Huygens, became familiar with the current issues in these fields. Among the problems and challenges that Huygens provided the young Leibniz was the determination of the sum of the reciprocals of triangular numbers.

Recall that triangular numbers are numbers that correspond to triangular arrangement of points as shown below:



1 1+2=**3** 1+2+3=**6** 1+2+3+4=**10**

It is known that the nth (general) triangular number T_n can be expressed by the formula:

$$T_n = 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$$

Leibniz supposedly solved this problem with a brilliant trick by realizing that

$$\sum_{n=1}^{\infty} \frac{1}{T_n} = \frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + ...\frac{1}{[n(n+1)/2]} + ...$$

$$= \sum_{n=1}^{\infty} \frac{1}{[n(n+1)/2]} = \sum_{n=1}^{\infty} \frac{2}{n(n+1)} = 2\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 2\sum_{n=1}^{\infty} (\frac{1}{n} - \frac{1}{(n+1)})$$

$$= 2[(\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{4} - \frac{1}{5}) + (\frac{1}{5} - \frac{1}{6}) + ...]$$

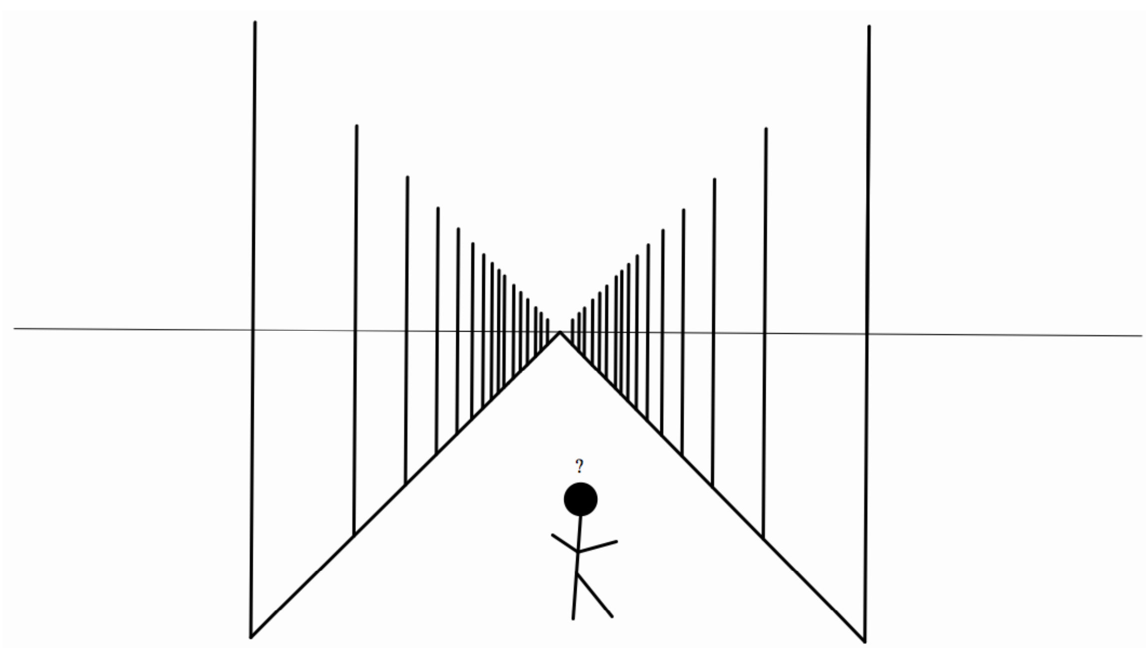
$$= 2[\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \frac{1}{5} - ....]$$

$$= 2$$

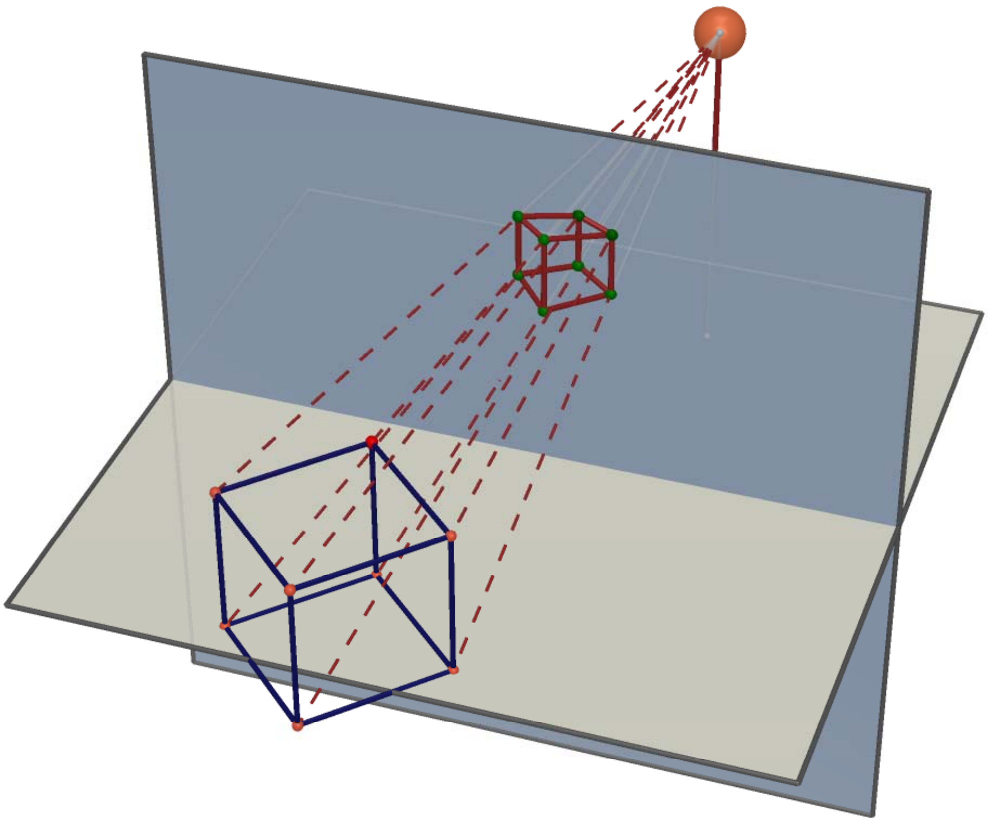
Though this casual treatment of an infinite series was later recognized to be potentially problematic, nonetheless it marked the beginnings of the unveiling of a great mathematical genius. (See William Dunham's wonderful book Journey Through Genius p184-190 for a more thorough treatment of this topic)

While teaching a chapter on Perspective Drawing with Michael Serra's Discovering Geometry textbook, at Istanbul American Robert College, I came to ask my students the following question: Let's say we are going to make a perspective drawing and wish to portray a typical scene with a road leading to the horizon with lampposts placed, say, everyone unit of distance apart. Having drawn the first post how do you decide where to place the second, the third … the n^{th} post on your drawing? How do you decide how to space them in your picture?

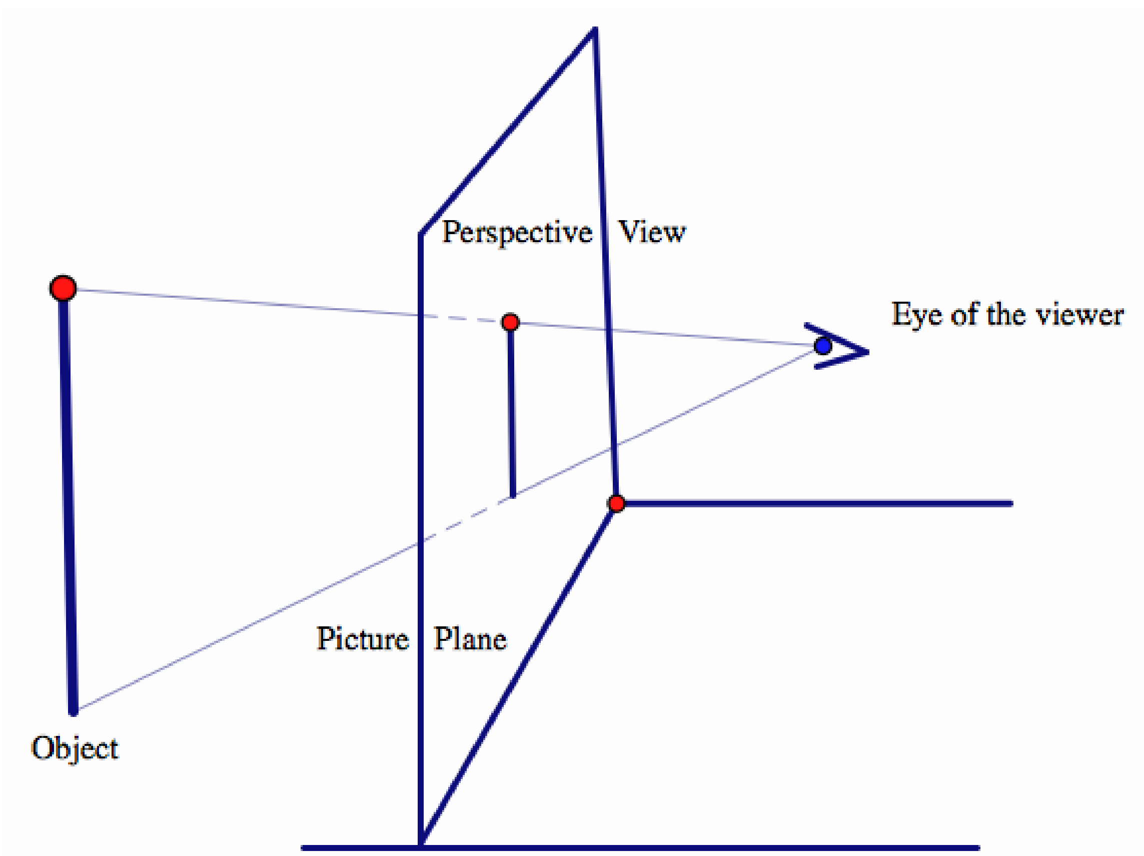
Three of my students Burak Cendek, Berkin Ozisikyilmaz and Kerem Celikel volunteered to be my research collaborators and trying to answer our original question we tried a specific case and came up with a construction scheme for reciprocal triangular numbers, which led to a nice geometric verification of Leibniz's above-mentioned proof that their sum is 2.

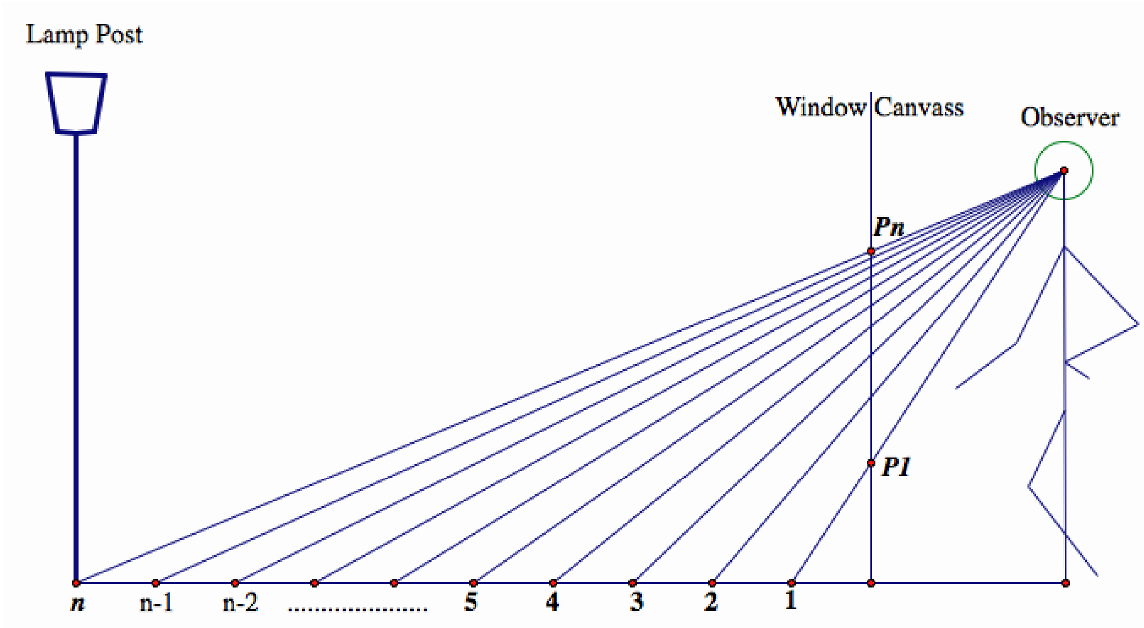


During our investigations we came to realize that we do not even understand the most basic question of the study of perspective: *why objects at a distance appear smaller*? Encyclopedia Britannica came to our rescue and defined the main idea of perspective drawing as, "*If while standing before a closed window you traced on the glass the outline of an object beyond, seen thorough it, you drew on the glass a true* ***picture or perspective*** *of that object. What took place in this process of tracing? From every visible point to the object there emanated a ray of light termed a* ***visual ray****, which pierced the glass and entered the eye. The outline you marked was the aggregate of points through which the rays of light from the object pierced the glass.*" This definition allowed us to make progress.

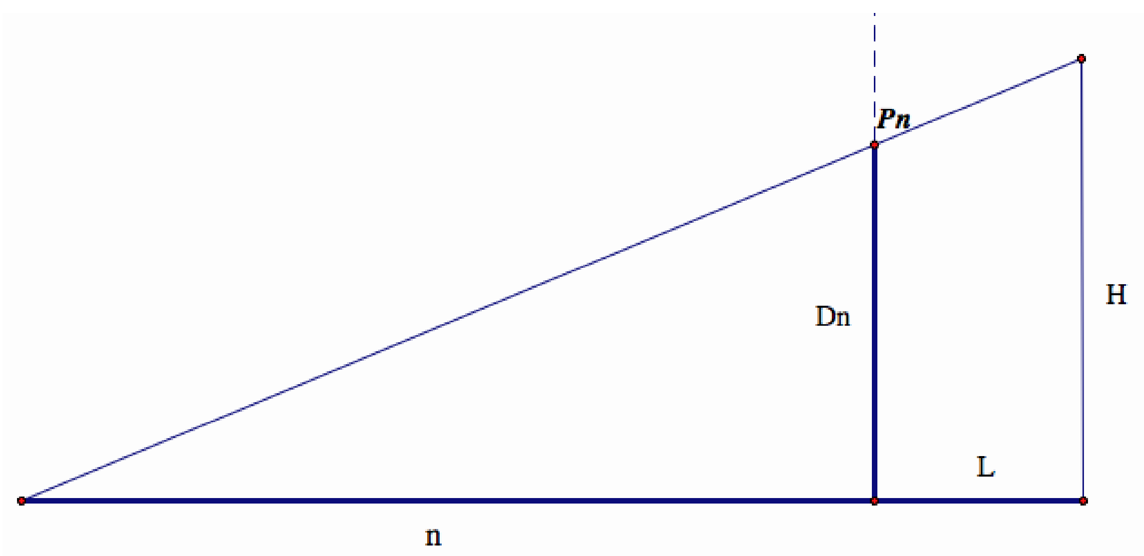


The problem of perspective at hand with the lampposts can be simplified if we imagine the artist looking at the actual scene behind a window canvass and recording/mapping the images of the objects on this window surface. So the problem posed above can be simplified with the following 2 dimensional side view of the situation.(Note that for further simplification we will assume that the artist looks at the posts head-on instead of sideways)





Positions 1,2,3,4,5,.., n-1, n represent the actual locations of the bottoms of the lampposts separated by unit distances. The light rays from these locations reach the artist's eye and allow him to see the bottoms of the lampposts. The points at which these rays intersect the window canvass (P_1,P_2,P_3,......P_n) are recorded by the artist and these will be the locations of the bottoms of the lampposts in the drawing. So the original question of how to place the posts on the drawing now reduces to deriving an expression for locating P_n , and studying the distance between consecutive points P_n and P_{n-1}.



If we let the eye level height of the artist be H, his distance from the canvass be L and let D_n represent the distance of point P_n from the ground, using similar triangles we get

$\frac{D_n}{n} = \frac{H}{L+n}$ and thus via cross multiplication $D_n = \frac{H \cdot n}{L+n}$

We can check that this equation makes sense as follows. As n approaches infinity we expect D_n to approach H, the eye level of the artist. A lamppost very far away will appear near the horizon line i.e. the eye level of the viewer. A simple limit calculation verifies this:

$$\lim_{n\to\infty} D_n = \lim_{n\to\infty} \frac{H \cdot n}{L+n} = \lim_{n\to\infty} \frac{H}{(L/n)+1} = H$$

At this point by chance we discovered the connection of this problem to the Leibniz problem previously mentioned. We wanted to get a feeling for our equation and when we let H=2 and L=l, (not unreasonable numbers in meters for the artist problem at hand) we get:

$$D_n = \frac{2 \cdot n}{1+n} = \frac{2n}{n+1}$$

Because in the original problem we will be interested in knowing the distance between consecutive points on the canvass, we need an expression for $D_n - D_{n-1}$ .We find D_{n-1} by substituting n-l for n in the above equation:

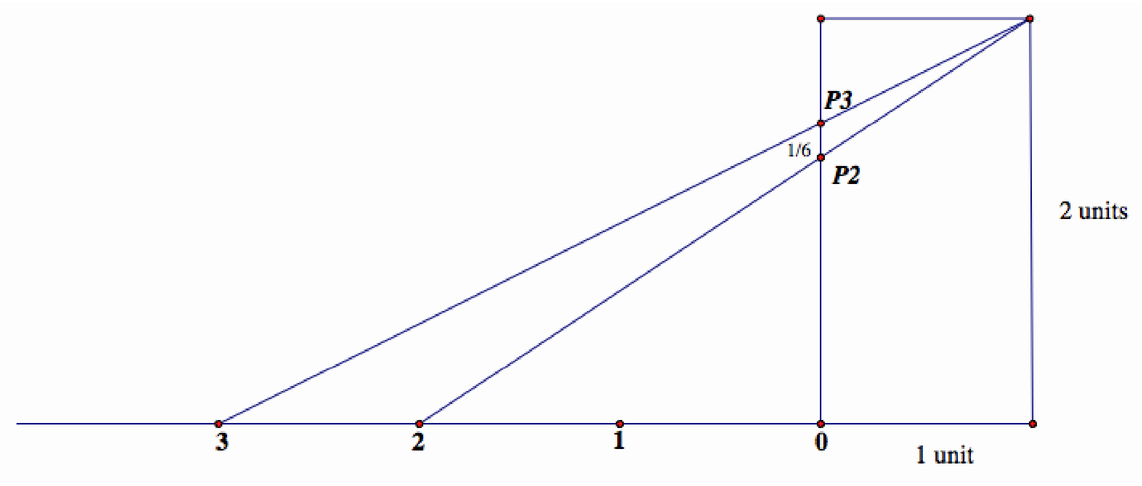
$$D_{n-1} = \frac{2(n-1)}{(n-1)+1} = \frac{2(n-1)}{n}$$

Now the surprising connection! The difference of these two quantities gives us:

$$D_n - D_{n-1} = \frac{2n}{n+1} - \frac{2(n-1)}{n} = \frac{2n^2 - 2(n-1)(n+1)}{n(n+1)} = \frac{2n^2 - 2n^2 + 2}{n(n+1)} = \frac{2}{n(n+1)}$$

This is the formula for the reciprocal of the n^{th} triangular number! So by choosing special values of H and L in our original problem we found a way to geometrically express and construct reciprocal triangular numbers.

Here is an example: Lets say we wanted to construct the third reciprocal triangular number, which is 1/6. We do it as shown in the diagram.



Steps:
(1) Construct a rectangle with width and length respectively 1 unit by 2 units, as shown.
(2) Extend the width of the rectangle with a ray and mark points that are 1, 2, 3, etc units away from the left bottom corner vertex of the rectangle (marked here as "0").
(3) Connect the 2^{nd} and 3^{rd} points to the top right corner vertex of the rectangle with segments.
(4) Mark the locations where these segments meet the left side of the rectangle (here indicated as P_2 and P_3). The length of the segment between these points is

$$D_3 - D_2 = \frac{2}{3(3+1)} = \frac{2}{12} = \frac{1}{6}$$

Note: To construct the reciprocal of an arbitrary triangular number T_n, create n points in step (2) and connect the "n-1^{th} and n^{th} points" to the top right corner vertex of the rectangle with segments in step (3).

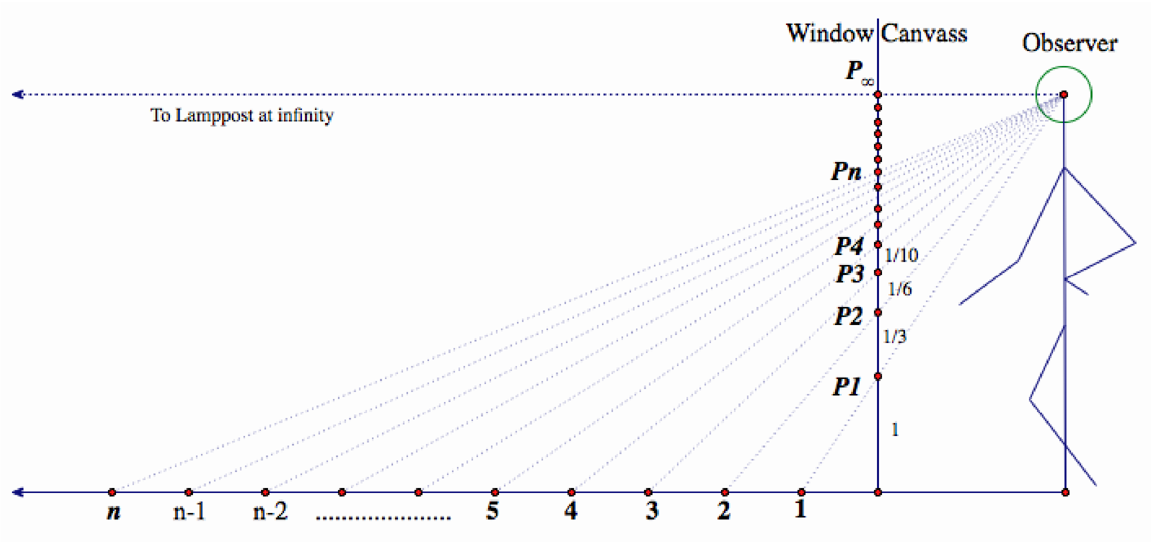
Please observe that the construction also provides a nice demonstration of Leibniz's proof mentioned earlier that:

$$\sum_{n=1}^{\infty} \frac{1}{Tn} = \frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + ... = 2$$

This is so because D_n, (for H=2 and L=1) which represents the n^{th} partial sum in the summation of reciprocal triangular numbers, has the limit:

$$\lim_{n\to\infty} D_n = \lim_{n\to\infty} \frac{H \cdot n}{L+n} = \lim_{n\to\infty} \frac{2}{(1/n)+1} = 2$$

This geometrically means that the observer (of eye height 2 units and 1 unit away from his canvass) will draw on his canvass a lamppost located very far (at infinity) near the horizon line i.e. his eye level. That objects which are very far are drawn near the horizon line is a well known simple fact in perspective studies.



Thus

$$P_\infty = 1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + ... = 2$$

I am not sure if our discovery is common knowledge or not. I don't know if Leibniz knew of such a geometric construction of reciprocal triangular numbers and whether artists of the Renaissance during their investigations came upon it. Probably someone must have done this work before. Honestly, it does not really matter. As Bruce Joyce says " We have to reinvent the wheel every once in a while not because we need a lot of wheels; but because we need a lot of inventors." Our discovery was unknown to us prior to our investigations and it surely excited us. I am sharing our findings in case someone what may find something of value in it. We enjoyed it tremendously and felt a link with the great master Leibniz as well as with great perspective artists of the Renaissance.

I would like to thank Key Curriculum Press for publishing the inspiring text Discovering Geometry and developing the software Geometer's Sketchpad which helped us greatly during our investigations on this matter. I also should note that the “GLaD” construction of reciprocal whole numbers and reciprocal Fibonacci numbers published a while ago also inspired us and gave us encouragement in thinking that our work may be of some importance. I would like to thank the two students and the teacher of this work and hope that our work provides a next link in their work and give some further insight into what they initiated.

Finally, I would like to thank Brian Swan, my good friend and colleague from Istanbul American Robert College for his feed back on this work.